$\mathbb{A}^1$-enumerative geometry

Use $\mathbb{A}^1$-homotopy theory to do enumerative geometry over any field k

- $\mathbb{A}^1$-homotopy theory
  = homotopy theory on smooth varieties$/k$

  $\mathbb{A}^1$ plays the role of the interval

Morel's $\mathbb{A}^1$-degree: (analog of Brouwer degree)

$[S^n, S^n] \to \mathbb{Z}$

$$\left[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1} \right]_{\mathbb{A}^1} \to GW(k)$$

$\mathbb{A}^1$-homotopy classes of maps

$$\mathbb{P}^n/\mathbb{P}^{n-1} \to \mathbb{P}^n/\mathbb{P}^{n-1} = \text{"}\mathbb{A}^1\text{-sphere"}$$

$GW(k)$ = Grothendieck Witt ring of k

Rmk $\mathbb{P}^n(\mathbb{R})/\mathbb{P}^{n-1}(\mathbb{R}) \simeq S^n$

$GW(k)$ = Grothendieck-Witt ring of k
= group completion of semi-ring
of isometry classes of
non-degenerate bilinear symmetric
forms

generators: $\langle a \rangle \quad a \in k^*$
$\qquad\qquad (x,y) \mapsto axy$

relations: 1) $\langle a \rangle = \langle ab^2 \rangle$

2) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

3) $\langle a \rangle + \langle b \rangle = \langle ab(a+b) \rangle + \langle a+b \rangle$

(4) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$)

Ex: $\cdot$ $GW(\mathbb{C}) \underset{rk}{\cong} \mathbb{Z}$

$\cdot$ $GW(\mathbb{R}) \underset{(rk,\, sgn)}{\cong} \mathbb{Z} \times \mathbb{Z}$

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

$sgn = \#1 - \#-1$

$\cdot$ $GW(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$
$\uparrow$
$(rk,\ disc)$
$\shortparallel$
det matrix

Kass-Wickelgren:

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \to GW(k)$$

$$f \in \qquad y \in$$

$$\deg^{\mathbb{A}^1} = \sum_{x \in f^{-1}(y)} \deg_x^{\mathbb{A}^1} f$$

U Nis nbhd of x

V — " — y

$$U/U-\{x\} \xrightarrow{} V/V-\{y\}$$

↑ induced by f

Watch orientations $\cong$ $\qquad$ $\cong$ Watch orientations

$$\mathbb{A}^n/\mathbb{A}^n-\{x\} \xrightarrow{(f_1,\dots,f_n)} \mathbb{A}^n/\mathbb{A}^n-\{y\}$$

$$\mathbb{P}^n/\mathbb{P}^{n-1} \cong \xrightarrow{\bar{f}} \cong \mathbb{P}^n/\mathbb{P}^{n-1}$$

$$\deg_x^{\mathbb{A}^1} f := \deg^{\mathbb{A}^1} \bar{f}$$

Computation:

locally

$(f_1,\dots,f_n): \mathbb{A}^n \to \mathbb{A}^n$

$$Jf(x) := \det \frac{\partial f_i}{\partial x_j}(x)$$

If x rational and $Jf(x) \neq 0$

$$\deg_x^{\mathbb{A}^1} f = \langle Jf(x) \rangle$$

Ex $k = \mathbb{R}$

Then for y a regular value

$$Jf(x) \in \{\pm 1\}$$

$\rightsquigarrow$ rotation of S^n or reflection

If x defined of L and $Jf(x) \neq 0$

Then $\deg_x^{\mathbb{A}^1} f = \mathrm{Tr}_{L/k}(\langle Jf(x) \rangle)$

$$= L \times L \xrightarrow{\uparrow} L \xrightarrow[\mathrm{Tr}_{L/k}]{} k$$

$Jf(x) : (a,b) \mapsto Jf(x)ab$

$\mathbb{A}^1$-Euler class (Kass–Wickelgren)

$\pi : E \to X$ oriented vector bundle

X ~~compact~~ proper and smooth mfld

$\operatorname{rk} E = \dim X = n$

Then
$$e^{\mathbb{A}^1}\left(\begin{matrix} E \\ \downarrow \\ X \end{matrix}\right) = \sum_{x \in \sigma^{-1}(0)} \deg_x^{\mathbb{A}^1} \sigma$$

for a section σ with only isolated zeros.

1^{st} example by Kass and Wickelgren:

lines on a cubic surface

$$X = \{F = 0\} \subseteq \mathbb{P}^3$$
F: general homogeneous of degree 3

$\rightsquigarrow$ section $G_F : Gr(2,4) \to \mathrm{Sym}^3 S^*$ by restriction
(with only isolated zeros)
$Gr(2,4)$ = lines in $\mathbb{P}^3$
S = tautological bundle

lines on X = zeros of section

$$\Rightarrow \sum_{x \in G_F^{-1}(0)} \deg_x^{\mathbb{A}^1} G_F = e^{\mathbb{A}^1}(\mathrm{Sym}^3 S^*)$$
$\leftarrow$ does not depend on choice of G

$\sum_{x \in G_F^{-1}(0)} \deg_x^{\mathbb{A}^1} G_F$ = count of lines on X

$$\underset{\text{Kass Wickelgren}}{=} 15\langle 1\rangle + 12\langle -1\rangle \in GW(k)$$

rk: 27 = complex count

sgn: 3 = real count = # hyperbolic lines − # elliptic lines

Lines on a Quintic 3-fold

$X = \{F = 0\} \subseteq \mathbb{P}^4$ (F: deg 5) $\rightsquigarrow$ section $G_F : Gr(2,5) \to \mathrm{Sym}^5 S^*$

$$e^{\mathbb{A}^1}(\mathrm{Sym}^5 S^*) = \sum_{x \in G_F^{-1}(0)} \deg_x^{\mathbb{A}^1} G_F = \text{count of lines on } X$$

Let $X = \{F = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subseteq \mathbb{P}^4$

be the Fermat quintic 3-fold.

<u>Problem</u>: There are infinitely many lines of the form $\{(s:-s:at:bt:ct)\}$ on X

$\Rightarrow$ σ_F does not have isolated zeros

<u>Fix</u>: Albano-Katz find 2875 distinguished lines on X which are the limits of lines on a general deformation

$$X_t = \{\underbrace{F + tF_1 + t^2F_2 + \dots}_{= F_t} = 0\} \subseteq \mathbb{P}^4$$

of X.

"<u>Dynamic Euler number</u>":

View X_t as a quintic 3-fold defined over $k((t))$ and compute $e^{\mathbb{A}^1}(\mathrm{Sym}^5 S^*_{k((t))}) \in GW(k((t)))$

Thm (P.)

$$\sum_{x \in G_{F_t}^{-1}(0)} \deg_{x_t} G_{F_t} = 1445\langle 1\rangle + 1430\langle -1\rangle \in GW(k((t)))$$

This is the image of $e^{\mathbb{A}^1}(\mathrm{Sym}^5 S_k^*)$ of $i : GW(k) \xrightarrow{\text{injective}} GW(k((t)))$

$$\langle a\rangle \mapsto \langle a\rangle$$

$$\Rightarrow \quad e^{\mathbb{A}^1}(\mathrm{Sym}^5 S^*) = 1445\langle 1\rangle + 1430\langle -1\rangle$$

<u>Rmk</u> Levine has already proved this using the theory of Witt-valued characteristic classes

Q: What information does this give us about the lines?

Cubic surface (Segre / $\mathbb{R}$, Kass-Wickelgren $\bar{k}$ char $k \neq 2$)

$\ell \subseteq X \subseteq \mathbb{P}^3$ defined over k

cubic surface $X = \{F = 0\} \subseteq \mathbb{P}^3$

Gauß map: $\mathbb{P}^1 \cong \ell \xrightarrow{\deg 2} \mathbb{P}^1 =$ 2-planes in $\mathbb{P}^3$ containing ℓ

$p \longmapsto T_p X$

This means to each $p \in \ell$ $\exists !\, q \in \ell$ st $T_p X = T_q X$

$\rightsquigarrow$ involution $i: \ell \to \ell$

$p \mapsto q$

fixed pts of i are defined over $k(\sqrt{\alpha})$ for $\alpha \in k^\times / k^{\times 2}$

Def $\operatorname{type}(\ell) := \langle a \rangle \in GW(k)$

Kass-Wickelgren: $\operatorname{type}(\ell) = \deg_{\ell} \sigma_F$

Ex (Segre) Over $\mathbb{R}$ there are 2 types called hyperbolic ($\langle +1 \rangle$) and elliptic $\langle -1 \rangle$ and # real hyperbolic lines $-$ # real elliptic ~~real~~ lines

$= 3$

3, 7, 15, 27

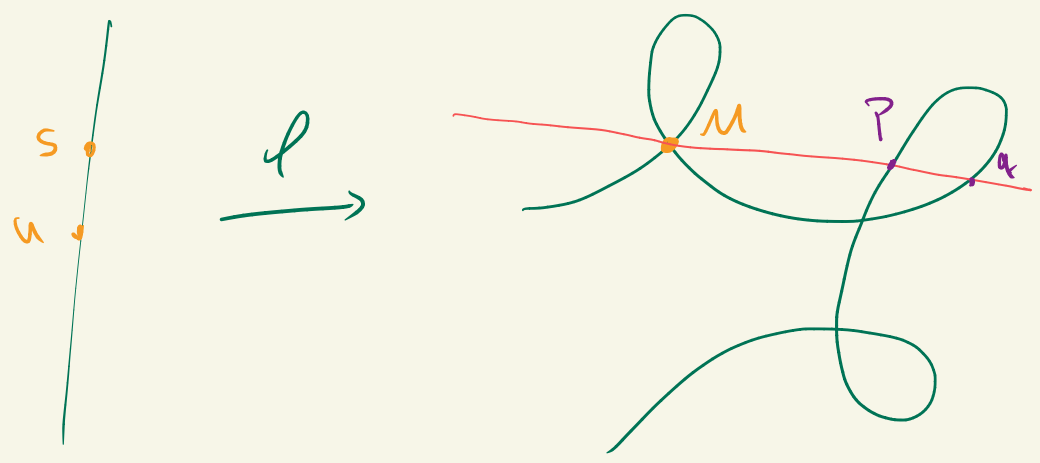
asu, stephen

Quintic 3-folds (Finashin-Kharlamov/$\mathbb{R}$, P./K)

$$\ell \subseteq X = \{\underset{\text{deg } 5}{F} = 0\} \subseteq \mathbb{P}^4$$

Gauß map $\ell \cong \mathbb{P}^1 \xrightarrow{\deg 4} \mathbb{P}^2 =$ 3-planes in $\mathbb{P}^4$ containing ℓ

$$p \longmapsto T_p X$$



There are 3 pairs of pts (s_j, u_j) $j=1,2,3$ on ℓ with

$$T_{s_j} X = T_{u_j} X$$

$\rightsquigarrow$ 3 involutions $\iota_j : \ell \to \ell$, $p \mapsto q$

p and q are st $\quad T := T_s X = T_u X$

$$T_p X \cap T_j = T_q X \cap T_j$$

Fixed pts of i_j are defined over

$$F_j(\sqrt{\alpha_j})$$

field of defn of M_j nodal pt

Def $\mathrm{Type}(\ell) = \langle \prod \alpha_j \rangle \in GW(k)$

Galois orbits of nodal pts

$F_j = k$ for $j = 1,2,3$

$= \langle \alpha_1 \alpha_2 \alpha_3 \rangle$

Thm (P.): $\mathrm{type}(\ell) = \deg_{[e]} {}^G F$

$\in GW(k)$

Macaulay 2 Examples

- Lines on a cubic surface as

$$e^{\mathbb{A}^1}\left(\begin{array}{c}\mathrm{Sym}^3 S^* \\ \downarrow \\ Gr(2,4)\end{array}\right) = 15\langle 1\rangle + 12\langle -1\rangle \in GW(k)$$

- lines meeting 3 general lines in $\mathbb{P}^3$ (Srinivasan, Wickelgren) as

$$e^{\mathbb{A}^1}\left(\begin{array}{c}\bigoplus^4 \Lambda^2 S^* \\ \downarrow \\ Gr(2,4)\end{array}\right) = \mathbb{H}$$

ask Kirsten for local contributions

l general line in $\mathbb{P}^3$ cut out by α, β linear form
$\rightsquigarrow \alpha \wedge \beta : Gr(2,4) \to \Lambda^2 S^*$ and a line l' meets l iff $\alpha\wedge\beta([l']) = 0$

- Lines on a degree 2 surface meeting 1 general line as

$$e^{\mathbb{A}^1}\left(\begin{array}{c}\mathrm{Sym}^2 S^* \oplus \Lambda^2 S^* \\ \downarrow \\ Gr(2,4)\end{array}\right) = 2\mathbb{H}$$

- Singular elements of a pencil of degree d surfaces $\{t_0F_0 + t_1F_1 = 0\} \subseteq \mathbb{P}^3 \times \mathbb{P}^1$ as

$$\overset{\rotatebox{90}{$=$}}{F_t}$$

$$e^{A^1}\left(\begin{array}{c} \bigoplus_{i=1}^{4} \pi_1^* \mathcal{O}_{\mathbb{P}^3}(d-1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^3}(1) \\ \downarrow \;\; \uparrow \; \frac{\partial F_t}{\partial x_1}, \ldots, \frac{\partial F_t}{\partial x_3} \\ \mathbb{P}^3 \times \mathbb{P}^1 \end{array}\right)$$

<u>Q</u> (Jesse Kass): • What is the contribution of singular surface?

- Same question for lines on deg 2 surface meeting a general line

$$\lambda \alpha \begin{pmatrix} a \\ b \end{pmatrix} = -\mu \alpha \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\lambda \beta \begin{pmatrix} a \\ b \end{pmatrix} = -\mu \beta \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\lambda = -\mu \frac{\beta \begin{pmatrix} c \\ d \end{pmatrix}}{\beta \begin{pmatrix} a \\ b \end{pmatrix}}$$

$$\mu \alpha \begin{pmatrix} c \\ d \end{pmatrix} = \mu \left(\alpha \begin{pmatrix} a \\ b \end{pmatrix} \cdot \frac{\beta \begin{pmatrix} c \\ d \end{pmatrix}}{\beta \begin{pmatrix} a \\ b \end{pmatrix}} \right)$$

$$\Rightarrow \quad \alpha \begin{pmatrix} c \\ d \end{pmatrix} \beta \begin{pmatrix} a \\ b \end{pmatrix} - \alpha \begin{pmatrix} a \\ b \end{pmatrix} \beta \begin{pmatrix} c \\ d \end{pmatrix} = 0$$

$$\Leftrightarrow \quad \alpha \wedge \beta (\ell) = 0$$

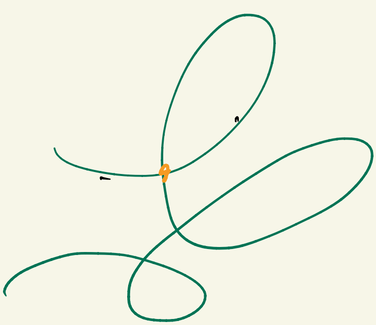
$C : \ell \xrightarrow{\deg 6} \mathbb{P}^3 =$ 4-planes in $\mathbb{P}^5$ containing ℓ

$$\ell \subseteq X = \{f = 0\} \subseteq \mathbb{P}^5$$

$\deg 7$

There are 6 lines in $\mathbb{P}^3$ meeting C in 4 pts $\ni s$

pencil of 2-planes containing s in $\mathbb{P}^3$ $\ni H$



$H \cap s = 6$ pts

$\{p_1, \dots, p_4\} \cup \{p, q\}$

$\in s$

$\longrightarrow \quad i : \ell \to \ell$

$$p \longmapsto q$$

fixed pts defined over

$L_S(\sqrt{\alpha})$

$$\mathrm{Type}(\ell) = \overline{\Pi}\,\alpha$$